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NATURAL OSCILLATIONS OF A LIQUID OF FINITE ELECTRICAL CONDUCTION IN THE PRESENCE OF AN EXTERNAL MAGNETIC FIELD

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The effect of the finite electrical conduction (finiteness of the magnetic Reynolds number), which is considered a dissipative factor, on small natural oscillations of an ideal heavy liquid of finite depth whose free surface borders on vacuum is studied. A constant external horizontal magnetic field is applied to the liquid. The energy-balance equation is derived, and the theorem of wave attenuation with time is proved. Numerical calculations and the resulting asymptotic formulas give a complete pattern of the spectrum, including its continuous part. The amplitude-frequency characteristics of the wave modes are presented.

In magnetohydrodynamics (MHD), the liquid is assumed to be an ideal liquid if it is nonviscous and infinitely conductive. The finiteness of the coefficient of conductivity is the dissipative factor associated with wave attenuation. In oscillation studies, two limiting cases of long and short waves are generally considered with the use of the model of an infinitely deep liquid having a free surface as a short-wave approximation. The investigations of the effect of viscosity on free surface waves, which were begun by Lamb [1] at the end of the 19th century, have acquired a complete mathematical form [2]. The theory of MHD waves in a conducting liquid is still far from a complete theory. A number of studies have been focused on this subject (see, e.g. [3–7]) where the spectrum of free oscillations of dissipative MHD waves in the canonical regions was not treated in detail. In the present paper, this spectrum for an infinitely deep liquid is studied by analytical and numerical methods. The discrete and continuous oscillation spectra of a heavy liquid of finite conduction subjected to an external horizontal magnetic field are considered.

1. Formulation of the Problem. We consider a nonviscous conducting liquid that occupies the lower half-space. There is a vacuum above the liquid. We introduce a Cartesian coordinate system such that the Oxy plane coincides with the undisturbed horizontal surface of the liquid, and the z axis is directed downward. Let the gravity $(0, 0, g)$ and a constant magnetic field $(H_0, 0, 0)$ be applied to the liquid. We investigate two-dimensional natural oscillations of the liquid in the xz plane. The liquid motion and the electromagnetic field are described by the equations given, for example, in [8]. We divide the space into two regions:

1. Liquid ($z \geqslant 0$). Let $\boldsymbol{V}(V_x, 0, V_z)$ be the velocity vector, ρ be the density, σ be the electrical conduction, and $\boldsymbol{h}(h_x, 0, h_z)$ and $\boldsymbol{e}(0, e_y, 0)$ be, respectively, the disturbances in the magnetic- and electric-field intensities caused by the liquid motion. Assuming that the oscillations are small, we write the linearized momentum and induction equations [8] in dimensionless form

$$\frac{\partial \boldsymbol{V}}{\partial t} = -\mathrm{grad}\,(p^*) + \frac{\partial \boldsymbol{h}}{\partial x}, \quad \mathrm{div}\,\boldsymbol{V} = 0, \quad \mathrm{div}\,\boldsymbol{h} = 0, \quad \frac{\partial h_x}{\partial t} = -\frac{\partial V_z}{\partial z} + \frac{1}{\mathrm{Re}_m}\Delta h_x, \tag{1}$$

$$\frac{\partial h_z}{\partial t} = -\frac{\partial V_z}{\partial x} + \frac{1}{\mathrm{Re}_m}\Delta h_z, \quad e_y = \frac{1}{c}\Big[\frac{1}{\mathrm{Re}_m}\Big(\frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x}\Big) - V_z\Big], \quad p^* = p_a + \frac{z}{\mathrm{Al}} + p_d + h_x,$$

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where $p_a = \text{const}$ is the free surface pressure, p_d is the hydrodynamic pressure, and Δ is the Laplace operator.

2. Vacuum ($z < 0$). We denote the disturbances in the electromagnetic-field intensity by $\boldsymbol{h}_1(h_{1x}, 0, h_{1z})$ and $\boldsymbol{e}_1(e_{1x}, e_{1y}, e_{1z})$. These functions satisfy the Maxwell equations (the displacement current is ignored)

$$\operatorname{rot}\boldsymbol{h}_1 = 0, \quad \operatorname{div}\boldsymbol{h}_1 = 0, \quad \frac{\partial e_{1y}}{\partial z} = \frac{1}{c}\frac{\partial h_{1x}}{\partial t}, \quad \frac{\partial e_{1y}}{\partial x} = -\frac{1}{c}\frac{dh_{1z}}{\partial t},$$
$$\frac{\partial e_{1x}}{\partial z} - \frac{\partial e_{1z}}{\partial x} = 0, \quad \frac{\partial e_{1x}}{\partial x} + \frac{\partial e_{1z}}{\partial z} = 0. \tag{2}$$

In Eqs. (1) and (2), $c = c_\nu\sqrt{4\pi\rho}/H_0$ is the velocity, c_ν is the dimensional velocity of light, $\nu_m = c_\nu^2/(4\pi\sigma)$ is the magnetic viscosity, $\text{Re}_m = LH_0/(\nu_m\sqrt{4\pi\rho})$ is the magnetic Reynolds number, $\text{Al} = H_0^2/(4\pi\rho gL)$ is the Alfvén number, $L = \Lambda/(2\pi)$ (Λ is the wavelength), and $L\sqrt{4\pi\rho}/H_0$ is the scale of time.

Let $z = \zeta(x, t)$ be the equation of a disturbed free surface (FS). On the FS, we introduce the linearized normal (subscript n) and tangential (subscript τ) components of the magnetic-field intensity vector $\boldsymbol{H}$ and tensor T and the total-stress tensor P:

$$H_n = h_z - \frac{\partial\zeta}{\partial x}, \quad H_\tau = 1 + h_x, \quad T_{nn} = -0.5 - h_x, \quad T_{n\tau} = h_z - \frac{\partial\zeta}{\partial x},$$
$$P_{nn} = -p + T_{nn}, \qquad P_{n\tau} = T_{n\tau}, \quad p = p_a + z/\text{Al} + p_d.$$

We specify the following boundary conditions for systems (1) and (2):

1) on the FS (liquid–vacuum interface) for $z = 0$:

$$V_z = \frac{\partial\zeta}{\partial t}, \qquad [H_n] = 0 \to h_z = h_{1z},$$
$$[e_\tau] = 0 \to e_y = e_{1y} = \frac{1}{c}\Big[\frac{1}{\text{Re}_m}\Big(\frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x}\Big) - V_z\Big], \quad e_x = e_{1x} = 0, \tag{3}$$
$$[P_{nn}] = 0 \to p^* + \zeta/\text{Al} = h_{1x}, \qquad [T_{n\tau}] = 0 \to h_x - h_{1x} = 0;$$

2) at infinity as $|z| \to \infty$. Two cases are possible here:

a) the fields attenuate in the liquid (discrete spectrum);

b) the fields are bounded in the liquid (continuous spectrum).

The necessity of considering the continuous spectrum stems from the fact that the solutions obtained within the framework of the model of an infinitely deep layer also make sense for a layer of finite depth.

2. Reduction to Boundary-Value Problems for Ordinary Differential Equations (ODE), Dissipation Theorem. To study natural oscillations, we separate the variables. Let

$$(V_x, V_z, h_x, h_z, e_x, e_y, e_z, p^*, \zeta) = (U, W, X, Z, E_x, E_y, E_z, Q, S)\exp(-\lambda t + ix),$$

where λ is the desired spectral parameter, whose real part is a damping constant and whose imaginary part is the phase velocity of a travelling wave. A similar notation is introduced for the vacuum (the subscript 1 is introduced). Substituting the last expression into (1) and (2), we obtain the following systems of ODE:

1. For the liquid,

$$-\lambda U(z) = -iQ(z) + iX(z), \qquad -\lambda W(z) = -Q'(z) + iZ(z),$$
$$-\lambda X(z) = -W'(z) + [X''(z) - X(z)]/\text{Re}_m, \qquad iX(z) + Z'(z) = 0,$$
$$-\lambda Z(z) = -W(z) + [Z''(z) - Z(z)]/\text{Re}_m, \qquad iU(z) + W'(z) = 0, \tag{4}$$
$$E_y = \frac{1}{c}\Big[\frac{1}{\text{Re}_m}(X'(z) - iZ(z)) - W(z)\Big];$$

2. For vacuum,

$$X_1'(z) - iZ_1(z) = 0, \quad iX_1(z) + Z_1'(z) = 0, \qquad E_{1y}'(z) = -(\lambda/c)X_1(z), \quad E_{1x}'(z) = iE_{1z}(z),$$

$$iE_{1y}(z) = (\lambda/c)Z_1(z), \qquad iE_{1x}(z) + E'_{1z}(z) = 0.$$

The boundary conditions (3) on the FS can be transformed to give

$$\begin{gathered} Q(0) + S/\mathrm{Al} = X_1(0), \quad X(0) = X_1(0), \quad Z(0) = Z_1(0), \quad W(0) = -\lambda S, \\ E_{1x}(0) = 0, \qquad E_{1y}(0) = \frac{1}{c}\Big[\frac{1}{\mathrm{Re}_m}(X'(0) - iZ(0)) - W(0)\Big]. \end{gathered} \tag{5}$$

The vacuum problem is readily solved. With allowance for wave attenuation at infinity ($z \to \infty$), we have

$$Z_1(z) = Z(0)\exp(z), \qquad X_1(z) = iZ(0)\exp(z),$$

$$E_{1y}(z) = -i(\lambda/c)\,Z(0)\exp(z), \qquad E_{1x}(z) = 0, \qquad E_{1z}(z) = 0.$$

We consider system (4) for the liquid. Additional rearrangements of conditions (5) on the FS yield

$$W(0) + \lambda S = 0, \qquad Q(0) + S/\mathrm{Al} = iZ(0), \qquad X(0) = iZ(0). \tag{6}$$

With allowance for (4) and (6), we can prove the following dissipation theorem.

Theorem. *All oscillations of the liquid attenuate for* $0 < \mathrm{Re}_m < \infty$ *and* $\mathrm{Al} > 0$, *i.e., the eigenvalues of the discrete spectrum lie in the right complex half-plane* [$\mathrm{Re}(\lambda) > 0$].

We multiply the first four equations of system (4) by complex-conjugate quantities and integrate over z from 0 to ∞. Then, taking into account the attenuation conditions at infinity, we integrate by parts and combine the expressions obtained. As a result, we have the energy-balance equation

$$\lambda(\|U\|^2 + \|W\|^2) + \bar\lambda\Big(\|X\|^2 + \|Z\|^2\Big) = \Big(\frac{1}{\mathrm{Re}_m} - \bar\lambda\Big)|Z(0)|^2$$

$$+ \frac{1}{\mathrm{Re}_m}(\bar X'(0)X(0) + \bar Z'(0)Z(0) - \bar Z''(0)Z(0)) + \frac{1}{\mathrm{Re}_m}(\|X'\|^2 + \|Z'\|^2 + \|X\|^2 + \|Z\|^2).$$

Here the norm can be interpreted as $\|F\|^2 = \int\limits_0^\infty |F(z)|^2 dz$ and the bar atop denotes the complex conjugation.

Furthermore, with allowance for the boundary conditions (6) on the FS, we obtain

$$\lambda(\|U\|^2 + \|W\|^2) + \bar\lambda(\|X\|^2 + \|Z\|^2 + |Z(0)|^2) + \frac{1}{\mathrm{Al}\,\lambda}|W(0)|^2$$

$$= \frac{2}{\mathrm{Re}_m}|Z(0)|^2 + \frac{1}{\mathrm{Re}_m}(\|X'\|^2 + \|Z'\|^2 + \|X\|^2 + \|Z\|^2).$$

We set $\lambda = \alpha + i\beta$ and separate the real and imaginary parts:

$$\alpha\{\|U\|^2 + \|W\|^2 + (\|X\|^2 + \|Z\|^2 + |Z(0)|^2)\} + \frac{\alpha}{(\alpha^2 + \beta^2)\mathrm{Al}}|W(0)|^2$$

$$= \frac{2}{\mathrm{Re}_m}|Z(0)|^2 + \frac{1}{\mathrm{Re}_m}(\|X'\|^2 + \|Z'\|^2 + \|X\|^2 + \|Z\|^2),$$

$$\beta(\|U\|^2 + \|W\|^2) = \beta(\|X\|^2 + \|Z\|^2 + |Z(0)|^2) + \frac{\beta}{(\alpha^2 + \beta^2)\mathrm{Al}}|W(0)|^2.$$

Hence, $\alpha > 0$. The theorem is proved.

Let $\beta \neq 0$. After elimination of β, the damping constant is given by

$$\alpha(\|U\|^2 + \|W\|^2) = \frac{1}{\mathrm{Re}_m}\Big(|Z(0)|^2 + \frac{1}{2}(\|X'\|^2 + \|Z'\|^2 + \|X\|^2 + \|Z\|^2)\Big).$$

This formula is rewritten in the convenient form $\alpha = D/(4K)$. Using the formula for the Joule heat density (MHD approximation)

$$D_\tau = (\mathrm{rot}\,\boldsymbol{H})^2/\mathrm{Re}_m,$$

one can show that D is the total Joule heat and K is the total kinetic energy. This result gives a physical interpretation of the dissipation theorem.

After simple manipulations, expressions (4) and (5) reduce to the fourth-order ODE

$$Z^{(\mathrm{IV})}(z) - 2Z''(z) + Z(z) + \mathrm{Re}_m(\lambda + 1/\lambda)(Z''(z) - Z(z)) = 0 \tag{7}$$

subject to the boundary conditions

$$\begin{gathered} Z'''(0) - Z'(0) - \frac{1}{\mathrm{Al}\lambda^2}(Z''(0) - Z(0)) + \mathrm{Re}_m\Big(\lambda + \frac{1}{\lambda}\Big)Z'(0) - \mathrm{Re}_m \frac{1+\mathrm{Al}}{\mathrm{Al}\,\lambda} Z(0) = 0, \\ Z'(0) - Z(0) = 0. \end{gathered} \tag{8}$$

As $z \to \infty$, either the attenuation $Z = 0$ or the boundedness $|Z| < \infty$ condition is specified. The resulting boundary-value problem is nonlinear in the spectral parameter λ.

3. Liquid Medium of Infinite Conduction (Ideal Liquid). This case is of interest as a limiting case of a finite-conduction liquid for $\mathrm{Re}_m = \infty$. From (7) and (8) [the second condition in (8) can be ignored], it follows that

$$(\lambda + 1/\lambda)(Z''(z) - Z(z)) = 0, \qquad \mathrm{Al}\,(\lambda + 1/\lambda)Z'(0) - (1 + \mathrm{Al})Z(0)/\lambda = 0.$$

We consider two cases:

1. For $\lambda_0 = \pm i\,\sqrt{(1 + 2\mathrm{Al})/\mathrm{Al}}$ and $Z(z) = C\exp(-z)$ (decrease with depth), we have a surface wave that does not decay with time and propagates at the phase velocity λ_0 in the horizontal direction.

2. For $\lambda_0 = \pm i \;\; \forall \;\; |Z''(z) - Z(z)| < \infty$, $|Z'(0)| < \infty$, $Z(0) = 0$, and $W(0) = 0$, we have a progressive internal Alfvén wave of arbitrary shape and subject to the "rigid cap" condition on its free surface. In this case, the spectrum of the problem has the infinite algebraic multiplicity.

Below, we show that these are the limiting cases for a liquid of finite conduction for discrete and continuous spectra, respectively.

4. Liquid Medium of Finite Conduction. Continuous Spectrum. We consider a medium of finite conduction [boundary-value problem (7), (8) under the boundedness condition at infinity]. The characteristic equation for this medium has the form

$$(\mu^2 - 1)(\mu^2 - (1 - m^2)) = 0, \qquad m^2 = \mathrm{Re}_m(\lambda + 1/\lambda).$$

The requirement that the solution be bounded in depth yields the general solution

$$Z(z) = C_1 \exp(-z) + C_2 \sin(\eta z) + C_3 \cos(\eta z),$$

where η is an arbitrary real parameter. The conditions on the FS imply that

$$\begin{gathered} C_2 = \frac{\lambda^4\mathrm{Al} + \lambda^2 + 2}{(1 - \lambda^2\mathrm{Al})\eta} C_1, \quad C_3 = \frac{\lambda^2(\lambda^2\mathrm{Al}+1+2\mathrm{Al})}{(1 - \lambda^2\mathrm{Al})\mathrm{Al}} C_1, \quad \lambda^2\mathrm{Al} \neq 1, \quad \lambda = \alpha \pm \sqrt{\alpha^2 - 1}, \\ \alpha = \frac{1 + \eta^2}{2\mathrm{Re}_m}, \quad Z(z) = C_1\Big(\exp(-z) + \frac{\lambda^4\mathrm{Al} + \lambda^2 + 2}{(1 - \lambda^2\mathrm{Al})\eta}\sin(\eta z) + \frac{\lambda^2(\lambda^2\mathrm{Al} + 1 + 2\mathrm{Al})}{(1 - \lambda^2\mathrm{Al})}\cos(\eta z)\Big). \end{gathered} \tag{9}$$

According to the terminology of [1], the wave structure in depth is cellular (it becomes periodic with increase in depth). When $\lambda = 1/\sqrt{\mathrm{Al}}$ (the case $\lambda = -1/\sqrt{\mathrm{Al}}$) is impossible), we have the periodic wave structure

$$C_1 = 0, \quad C_3 = C_2/\eta, \qquad \eta^2 = \mathrm{Re}_m(1 + \mathrm{Al})/\sqrt{\mathrm{Al}} - 1,$$

$$Z(z) = C_2(\cos(\eta z) + \sin(\eta z)/\eta), \qquad \eta \neq 0, \qquad \mathrm{Re}_m > \sqrt{\mathrm{Al}}/(1 + \mathrm{Al}).$$

The following properties of the continuous spectrum are noteworthy:

1) for $\alpha < 1$, there exists a travelling wave that attenuates with time; in this case, $\alpha = (1 + \eta^2)/(2\mathrm{Re}_m)$, $\beta = \sqrt{1 - \alpha^2}$, and $|\lambda| = 1$ [the right half-circle in the complex plane λ; $\mathrm{Re}(\lambda) > 0$];

2) for $\alpha > 1$, two nonoscillating regimes (modes) occur, $\lambda_{\mathrm{slow}} = \alpha - \sqrt{\alpha^2 - 1}$ being the decrement of a slowly attenuating mode and $\lambda_{\mathrm{rapid}} = \alpha + \sqrt{\alpha^2 - 1}$ being the decrement of a rapidly attenuating mode;

3) a multiple real root $\lambda^* = 1$ exists;

4) the waves are internal in nature: the FS remains undeformed and the "rigid cap" condition $Z(0) = 0$ holds;

5) the magnetic field in the vacuum is not disturbed;

6) the limit solution as $\mathrm{Re}_m \to \infty$ is the internal Alfvén wave

$$\lambda = \pm i, \qquad Z(z) = C(\exp(-z) + \sin(\eta z)/\eta - \cos(\eta z)),$$

whose structure is determined by the arbitrary real parameter η. This result refines the findings of Sec. 3 for an infinite-conduction liquid.

Let us consider the case of a multiple root in detail, namely, the process of merging the real roots and the appearance of a pair of complex-conjugate roots [$\lambda^* = 1$ and $\mathrm{Re}_m^* = (1+\eta^2)/2$]. Separating the variables, we seek a solution in the form

$$\Phi(z,t) = \exp(-\lambda t) \sum \frac{\Phi_{k-m}(z) t^m}{m!}.$$

Substituting this expression into the equations of the initial boundary-value problem and equating the coefficients of the same powers of t yield a sequence of boundary-value problems for determining the functions $\Phi_j(z,\lambda)$. This sequence is truncated for $j = 2$, i.e., the sum contains only two terms and has the form

$$\Phi(z,t) = (tZ(z,\lambda) + Z'_\lambda(z,\lambda)) \exp(-\lambda t),$$

where the eigenfunction $Z(z,\lambda)$ is given by expression (9), and the associated function Z'_λ is determined by the derivative of the eigenfunction with respect to the spectral parameter λ. This is characteristic of holomorphic operator beams [2]. The time-dependent exponential factor in the above formula leads to a possible initial increase in the amplitude followed by the exponential damping.

5. Liquid of Finite Conduction. Discrete Spectrum. With the solution of the boundary-value problem (7), (8) subject to the attenuation-with-depth condition ($z \to \infty$), we have

$$Z(z) = C_1 \exp(-z) + C_2 \exp(-æ z), \quad æ = \sqrt{1 - \mathrm{Re}_m(\lambda + 1/\lambda)}, \quad \mathrm{Re}(æ) > 0, \quad \lambda \neq \pm i. \qquad (10)$$

The degenerate case $\lambda = \pm i$ corresponds to a root of multiplicity two $æ = 1$. Furthermore, it can be shown that an attenuation-with-depth regime is absent. For other values of λ, we satisfy the boundary conditions on the FS and obtain the dispersion equation

$$\sqrt{1 - \mathrm{Re}_m\Big(\lambda + \frac{1}{\lambda}\Big)} = -\frac{\lambda^4 \mathrm{Al} + \lambda^2 + 2}{\lambda^2(\lambda^2 \mathrm{Al} + 1 + 2\mathrm{Al})}. \qquad (11)$$

In this case, the relation $C_1 = -0.5(1 + æ)C_2$ is valid. Evidently, Eq. (11) has no real positive roots. It also has two extraneous roots $\lambda = \pm i$. After these roots are eliminated, we obtain

$$\sqrt{1 - \mathrm{Re}_m\Big(\lambda + \frac{1}{\lambda}\Big)} = \frac{\mathrm{Re}_m}{2} \frac{\lambda(\lambda^2 \mathrm{Al} + 1 + 2\mathrm{Al})}{\lambda^2 \mathrm{Al} + 1} - 1.$$

This equation is equivalent to the polynomial

$$P(\lambda, \mathrm{Al}, \mathrm{Re}_m) = \lambda^3(\lambda^2 \mathrm{Al} + 1 + 2\mathrm{Al})^2 - \frac{4}{\mathrm{Re}_m}(\lambda^4 \mathrm{Al}^2 - 1) = 0$$

with the condition for the choice of extraneous roots

$$\mathrm{Re}\Big(\mathrm{Re}_m \frac{\lambda(\lambda^2 \mathrm{Al} + 1 + 2\mathrm{Al})}{2(\lambda^2 \mathrm{Al} + 1)} - 1\Big) \geqslant 0.$$

Figure 1 shows the behavior of the dispersion equation in the complex plane $\lambda = \alpha + \beta i$ for fixed Alfvén numbers (Al varies from 0.7 to 1.9 with step 0.2). Two attenuating regimes of oscillations can exist, which we call high-frequency and low-frequency modes. As $\mathrm{Re}_m \to \infty$, for a high-frequency mode, we observe a continuous limiting passage to the case of a liquid of finite conduction, which was considered in Sec. 3. We write the corresponding asymptotic expansion

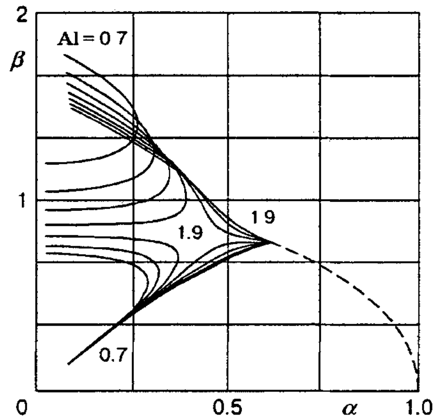

Fig. 1

$$\lambda = \pm i\sqrt{\frac{1+2\mathrm{Al}}{\mathrm{Al}}}\Big(1 - \sqrt{\frac{2}{\mathrm{Re}_m}}\,\frac{\mathrm{Al}^{5/4}(1+\mathrm{Al})^{1/2}}{(1+2\mathrm{Al})^{7/4}}\Big) + \sqrt{\frac{2}{\mathrm{Re}_m}}\,\frac{\mathrm{Al}^{3/4}(1+\mathrm{Al})^{1/2}}{(1+2\mathrm{Al})^{5/4}} + O\Big(\frac{1}{\mathrm{Re}_m}\Big).$$

It should be noted that the above and all subsequent asymptotic expansions are obtained with the use of the Newton diagram [9]. In the case of an infinite-conduction liquid (see Sec. 3), the low-frequency mode is absent. For $\mathrm{Al}/\mathrm{Re}_m \ll 1$, we have the asymptotic formula

$$\lambda = \sqrt[3]{\frac{4\,\mathrm{Al}}{\mathrm{Re}_m}}\Big(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\Big) + O\Big(\frac{\mathrm{Al}}{\mathrm{Re}_m}\Big)^{2/3}.$$

For $\mathrm{Al} < \mathrm{Al}^*$, the dispersive lines of the high-frequency mode have maximum points of the damping constant. The low-frequency mode possesses this property for $\mathrm{Al} > \mathrm{Al}^*$. It is noteworthy that the solutions have "break" points, which form a "break" curve shown as a dashed curve in Fig. 1. This complex behavior of the dispersive curves is attributed to the existence of the complex root of multiplicity two, which is determined from the system (bifurcation equations)

$$P(\lambda, \mathrm{Al}, \mathrm{Re}_m) = 0, \qquad P'_\lambda(\lambda, \mathrm{Al}, \mathrm{Re}_m) = 0.$$

An analysis shows that the system has the unique solution in the right half-plane $\mathrm{Re}(\lambda) > 0$ for $\mathrm{Al} > 0$ and $\mathrm{Re}_m > 0$

$$\lambda^* = 0.417498 \pm i \cdot 0.818077, \qquad \mathrm{Al}^* = 1.417677, \qquad \mathrm{Re}^*_m = 0.925158,$$

which was found numerically. We note that a complex root of multiplicity two rarely occurs in hydromechanics [2]; therefore, it is of considerable interest. In the neighborhood of this root, we have the asymptotic form

$$\lambda = \lambda^* \pm (0.418\,427 \mp i \cdot 0.024\,987)\sqrt{\frac{1}{\mathrm{Re}_m} - \frac{1}{\mathrm{Re}^*_m}}.$$

Any branch of the root can be chosen. It follows that the difference between these roots decreases for $\mathrm{Re}_m > \mathrm{Re}^*_m$ and increases for $\mathrm{Re}_m < \mathrm{Re}^*_m$. As in the case of a multiple real root (see Sec. 4), the complex amplitude has the form

$$Z(z, t, \lambda^*) = (tZ(z, \lambda^*) + Z'_\lambda(z, \lambda^*))\exp(-\lambda^* t).$$

The eigenfunction $Z(z, \lambda)$ is given by formula (10). The time-dependent factor t can also lead to an initial increase in the wave amplitude followed by the exponential damping.

The "break" curve is determined by solving numerically the system

$$P(\lambda, \mathrm{Al}, \mathrm{Re}_m) = 0, \qquad \mathrm{Re}\Big(\mathrm{Re}_m\,\frac{\lambda(\lambda^2\mathrm{Al} + 1 + 2\mathrm{Al})}{2(\lambda^2\mathrm{Al} + 1)} - 1\Big) = 0.$$

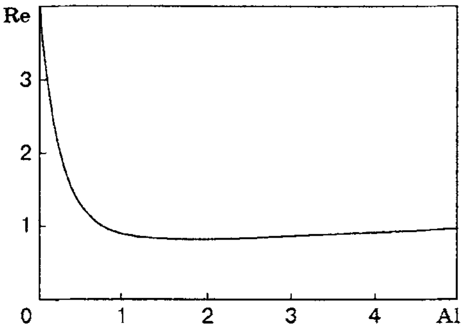

Fig. 2

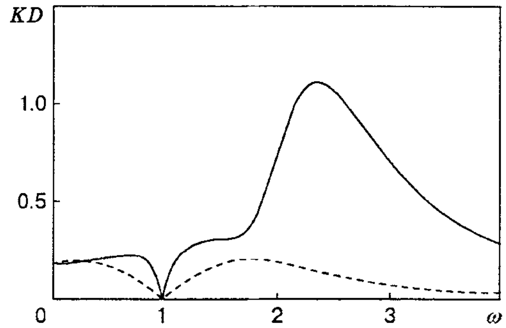

Fig. 3

As a result, we obtain the universal relationship $\mathrm{Re}_m(\mathrm{Al})$ plotted in Fig. 2. The graph has the horizontal asymptote $\mathrm{Re}_m = 4/3$ as $\mathrm{Al} \to \infty$. In the region above the graph, there are two (high-frequency and low-frequency) oscillation regimes that attenuate with time. In the region below the graph, the high-frequency ($\mathrm{Al} > \mathrm{Al}^*$) and low-frequency ($\mathrm{Al} < \mathrm{Al}^*$) regimes occur, respectively, on the right and left parts of the region. A numerical analysis shows that on the "break" curve, the exponent $æ$ in formula (10) vanishes and the eigenfunction does not decay with depth. With further decrease in the magnetic Reynolds number Re_m, the discrete spectrum becomes a continuous spectrum, and the dispersive curve is shaped like tan arc of a circle of unit radius up to the multiple point $\lambda = 1$ (the dashed curve in Fig. 1), then the regimes of slow and rapid aperiodic damping occur (see Sec. 4).

In experimental oscillation studies, it is of interest to consider the amplitude-frequency characteristics that correspond to a monochromatic wave of excitation pressure $p = p_0 \cos(x + \omega t)$ that travels over the FS [$\omega \in [0, \infty)$ and $\mathrm{Im}(p_0) = 0$]. Figure 3 shows the dynamic coefficient KD for the amplitude of vertical magnetic intensity versus the parameter ω for $\mathrm{Al} = 0.5$ (the solid curve refers to $\mathrm{Re}_m = 5$, and the dashed curve to $\mathrm{Re}_m = 1$). As should be expected, the quasiresonant burst is pronounced for sufficiently large magnetic Reynolds numbers and travelling-wave velocities close to the spectral velocities: $\beta = \mathrm{Im}(\lambda)$.

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